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ON PROPAGATION OF SHOCK WAVES IN AN ELASTIC MEDIUM UNDER PLANE FINITE STRAIN

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Shock wave propagation conditions in an elastic medium with a Murnaghan potential [1] are investigated. The velocities of the possible shock waves are found from the solution of the system of equations in jumps on the shocks. Necessary conditions for the existence of shocks, analogous to the Zemplen theorem for a perfect gas, are obtained by using the second law of thermodynamics on the surface of discontinuity.

The propagation of weak waves of discontinuities in an elastic medium under finite strains has been investigated in [2] et al. The elastic medium was considered incompressible in [3]. Shock waves propagation has been studied for a compressible elastic medium in [4 – 6], etc. The governing equations were written as a generalized Hooke's law in [4]; expressions were obtained for the shock velocities as was the existence condition for a quasi-transverse shock. The shocks were studied for the particular cases of deformation of the medium ahead of the shock [5, 6].

1. Let us define an elastic medium by the Murnaghan potential

$$W = aI_1^2 + cI_2 + lI_1I_2 + mI_3 + nI_1^3 \tag{1.1}$$
$$I_1 = e_{kk}, \qquad I_2 = e_{ki}e_{ki}, \qquad I_3 = e_{ik}e_{kj}e_{ji}$$
$$e_{ij} = {}^1/_2\,(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j})$$

Here e_{ij} is the Almansi finite strain tensor, I_1, I_2, I_3 are the invariants of this tensor, and u_i are the displacement vector components in a Cartesian coordinate system. The coefficients a and c are expressed linearly in terms of the Lamé parameters, and the coefficients l, m, n are the ordinary Murnaghan coefficients or third-order elastic moduli. The applicability of (1.1) for a broad class of materials is shown in [7, 8], where these coefficients have been determined experimentally.

The following constraints on the coefficients a, c, l, m, n in (1.1) result from the condition of nonnegativity of the function W :

$$a > 0,\ \ c > 0,\ \ \operatorname{sign} l = \operatorname{sign} n = \operatorname{sign} I_1 \tag{1.2}$$
$$\operatorname{sign} m = \operatorname{sign} I_3$$

The equality

$$I_3 = (2\,e_{12}^2 + e_{11}^2 + e_{22}^2 - e_{11}e_{22})\,I_1$$

is satisfied in the case of the plane state of strain. The expression in parentheses is nonnegative, hence

$$\operatorname{sign} I_1 = \operatorname{sign} I_3 = \operatorname{sign} m \tag{1.3}$$

Let us calculate the Euler strain tensor σ_{ij} from the following formula [5]:

$$\sigma_{ij} = \frac{\rho}{\rho_0} \frac{\partial W}{\partial e_{ik}} (\delta_{kj} - e_{kj}) \tag{1.4}$$

$$\frac{\rho}{\rho_0} = \left(1 - 2I_1 + 2I_1^2 - 2I_2 - \frac{4}{3} I_1^3 + 4I_1I_2 - \frac{8}{3} I_3\right)^{1/2}$$

The relative density ρ / ρ_0 is expressed in terms of the Almansi tensor invariants [1]. Substitutung (1.1) into (1.4) results in the governing equation

$$\sigma_{ij} = 2\, a e_{kk}\delta_{ij} + 2\, c e_{ij} + (3\, n - 2\, a)\, u_{k,k}^2\, \delta_{ij} + l v_{sk} v_{ks} \delta_{ij} + 2\,(l - 2\, a - c)\, u_{k,k} v_{ij} + (3\, m - 4\, c)\, v_{ik} v_{kj} \tag{1.5}$$

$$v_{ij} = {}^1/_2\, (u_{i,j} + u_{j,i})$$

Let a surface of discontinuity Σ move at some velocity G in the medium under consideration. Let us introduce a moving system of coordinates coupled to Σ, let us direct the x_2 axis normal to the line of discontinuity. Let the fixed coordinate system have axes parallel to the moving axes at some time, then the transformation formulas are

$$\frac{\partial}{\partial x_i} = \delta_{i2} \frac{\partial}{\partial x_2} + \delta_{i1} \frac{\partial}{\partial x_1}, \qquad \frac{\partial}{\partial t} = \frac{\delta}{\delta t} - G \frac{\partial}{\partial x_2}$$

Here $\delta / \delta t$ is the delta derivative with respect to time [9]. The displacements on Σ are continuous, hence

$$[u_{i,j}] = [u_{i,2}]\, \delta_{j2} \tag{1.6}$$

The square brackets denote jumps in the corresponding quantities. The displacement velocities in the moving coordinate system are calculated from the formula

$$v_i = \delta u_i / \delta t + (v_2 - G)\, u_{i,2} + v_1 u_{i,1} \tag{1.7}$$

Performing the jump operation in (1.7) and taking account of (1.6), we obtain

$$[v_i] = u_{i,1}\, [v_1] + [(v_2 - G)\, u_{i,2}] \tag{1.8}$$

Let us examine the particular case when the displacement vector components ahead of the shock are independent of x_1, then from the expression for e_{ij} in (1.1), (1.5) and (1.6) we obtain expressions for the jumps in the strain and stress tensors for the case of the plane state of strain

$$[\sigma_{12}] = f_{1k}\, [u_{k,2}], \qquad [\sigma_{22}] = f_{2k}\, [u_{k,2}] \tag{1.9}$$

$$f_{11} = \{c - \lambda_1\, (u_{2,2}^+ - [u_{2,2}])\}, \qquad f_{12} = -\lambda_1 u_{1,2}^+$$

$$f_{21} = -\lambda_3\, (2u_{2,2}^+ - [u_{2,2}]), \qquad f_{22} = \lambda_0 - \lambda_2\, (2u_{2,2}^+ - [u_{2,2}])$$

$$\lambda_0 = 2\,(a + c), \qquad \lambda_1 = 2a + 3c - l - {}^3/_2\, m$$

$$\lambda_2 = 7\,(a + c) - 3\,(l + m + n), \qquad \lambda_3 = {}^1/_4\, (4a + 8c - 2l - 3m)$$

Solving (1.8) for $[v_i]$, we obtain

$$[v_1] = (v_2^- - G)\{[u_{1,2}] + u_{1,2}^+\, [u_{2,2}] / (1 - u_{2,2}^+)\} \tag{1.10}$$

$$[v_2] = (v_2^- - G)\, [u_{2,2}] / (1 - u_{2,2}^+)$$

Let us append the dynamical comatibility conditions for the discontinuities on the shock to (1.9), (1.10)

$$[\sigma_{12}] = \rho^- (v_2^- - G)[v_1] \qquad (1.11)$$
$$[\sigma_{22}] = \rho^- (v_2^- - G)[v_2], \qquad [\rho (v_2 - G)] = 0$$

Equations (1.9) – (1.11) form a closed system in the jumps of the discontinuous quantities. Eliminating the jump in the stress and strain tensor components therefrom, we arrive at the following:

$$V \{[u_{1,2}] + u_{1,2}^+ [u_{2,2}] / (1 - u_{2,2}^+)\} = f_{1k} [u_{k,2}] \qquad (1.12)$$
$$V[u_{2,2}] = (1 - u_{2,2}^+) f_{2k} [u_{k,2}], \qquad V = \rho^- (v_2^- - G)^2$$

Let us consider V an unknown quantity characterizing the velocity of shock wave propagation. Eliminating $[u_{1,2}]$ from (1.12), we obtain a cubic equation in V

$$AV^3 - BV^2 - CV + D = 0 \qquad (1.13)$$

Because of unwieldiness we do not present expressions of the coefficients A, B, C, D in terms of the quantities in (1.12). If the discriminant of the cubic equation (1.13) is positive, then it has three real roots. Since $V > 0$, then the number of positive roots governs the number of possible shocks.

2. Let us use the small parameter method to investigate the roots of (1.13) by considering $u_{1,2}^+$ a small quantity. Neglecting squares of this quantity in (1.13), we obtain

$$A_0 V^3 + B_0 V^2 + C_0 V + D_0 = 0 \qquad (2.1)$$
$$A_0 = 1 - u_{2,2}^+, \qquad B_0 = A_0^2 f_{22} + 2 A_0 f_{11}$$
$$C_0 = A_0 f_{11}^2 + 2 A_0^2 f_{22} f_{11}, \qquad D_0 = A_0^2 f_{22} f_{11}^2$$

which can be reduced to

$$(V - V_1^\circ)(V - V_2^\circ)(V - V_3^\circ) = 0 \qquad (2.2)$$
$$V_1^\circ = A_0 f_{22}, \qquad V_2^\circ = V_3^\circ = f_{11}$$

Substituting $V = V_1^\circ$ and $V = V_2^\circ$ into (1.12) in the same approximation results in the respective equalities

$$[u_{1,2}] = 0, \qquad [u_{1,2}] = (f_{22} + f_{11} A_0^{-1}) [u_{2,2}] \qquad (2.3)$$

Hence, it is seen that the first approximation in the case considered yields the same results as the linear theory: two shocks, a longitudinal and a transverse, are possible. The nonlinearity is felt only quantitatively in the values for the propagation velocities for these shocks.

Let us turn to finding the second approximation for the roots of (1.13), which we shall represent as the sum of the first approximation and of a small additional term

$$V_1 = V_1^\circ + V_1^1, \qquad V_2 = V_2^\circ + V_2^1 \qquad (2.4)$$

Substituting (2.4) into (1.13), we obtain the following values for V_1 and V_2 to higher order accuracy

$$V_1 = V_1^\circ - \frac{\lambda_3 (V_1^\circ + \lambda_1 A_0)}{A_0 (V_1^\circ - V_2^\circ)} \{2A_0 (V_1^\circ - V_2^\circ) + V_1^\circ + \lambda_1 A_0 [u_{2,2}]\} u_{1,2}^2 \qquad (2.5)$$
$$V_{2,3} = V_2^\circ \pm (V_2^\circ + A_0 \lambda_1) \left\{ \frac{\lambda_3 [u_{2,2}]}{2A_0 (V_2^\circ - V_1^\circ)} \right\}^{1/2} u_{1,2}^+$$

Let us call the shock corresponding to the root V_1 quasi-longitudinal, and to the roots $V_{2,3}$ quasi-transverse. The existence of quasi-transverse waves is possible if the discriminant in the cubic equation (1.13) is positive resulting in the inequality

$$\lambda_3 [u_{2,2}] \leqslant 0 \tag{2.6}$$

For known elastic materials, the coefficients l, m, n are small compared to the coefficients a and c, i.e. $\lambda_3 > 0$, hence, from (2.6) we obtain the inequality

$$[u_{2,2}] \leqslant 0 \tag{2.7}$$

Let us note that the inequality (2.6) is valid for any l, m, n for $I_1 < 0$ according to (1.2) and (1.3). The inequality (2.7) was first obtained in [4] for a quasi-Hooke model of an elastic medium.

3. The thermodynamic condition of compatibility of the discontinuities, which is a corollary of the second law of thermodynamics, should be satisfied on the shock. Let us write this condition as [10]

$$\frac{1}{2} A [v_i v_i] + A_i [v_i] - \frac{A}{\rho_0} [W] \geqslant 0 \tag{3.1}$$

$$A = \rho^- (v_2^- - G), \qquad A_i = \sigma^+ - A v_i^+$$

In this case the inequality (3.1) becomes the following:

$$-\frac{1}{2} \rho^- (v_2^- - G) [v_1]^2 - \frac{1}{2} \rho^- (v_2 - G) [v_2]^2 + \tag{3.2}$$

$$\sigma_{12}^+ [v_1] + \sigma_{22}^+ [v_2] - \frac{\rho}{\rho_0} (v_2^- - G) [W] \geqslant 0$$

Using (1.1), the jump in the elastic potential can be represented as

$$[W] = a[(e_{11} + e_{22})^2] + (l + {}^3/_2\, m) [(e_{11} + e_{22})\, e_{ik} e_{ki}] + \tag{3.3}$$

$$c\, [e_{ik} e_{ki}] + (n - {}^1/_2\, m) [(e_{11} + e_{22})^3]$$

From (1.8) we find the values of the stress tensor components σ_{12}^+ and σ_{22}^+

$$\sigma_{12}^+ = c u_{1,2}^+ - \lambda_1 u_{1,2}^+ u_{2,2}^+, \qquad \sigma_{22}^+ = \lambda_0 u_{2,2}^+ - \lambda_2 u_{2,2}^{+2} - \lambda_3 u_{1,2}^{+2} \tag{3.4}$$

Let us examine the inequality (3.2) in the case of a quasi-longitudinal shock. Substituting (3.3), (3.4), (1.14) and the first equality from (2.4) into (3.2) and limiting ourselves to the cubes of the tensor components of the displacement gradient, we obtain the inequality

$$(a + c + l + m + n) [u_{2,2}] \geqslant 0 \tag{3.5}$$

For the case when the coefficients l, m and n are small compared to the coefficients a and c, the inequality (3.5) takes the simple form

$$[u_{2,2}] \geqslant 0 \tag{3.6}$$

It follows from (3.6) that quasi-longitudinal rarefaction shocks are impossiblw in elastic media. In the case of a quasi-transverse shock, the left side of inequality (3.2) is identically zero, to the accuracy of cubes in u_{ij} .

Therefore, if energy dissipation by a quasi-longitudinal shock is of third order in the tensor components $u_{i,j}$, then energy dissipation by a quasi-transverse wave is higher than third order in the case under consideration.

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EQUILIBRIUM OF A NONHOMOGENEOUS HALF-PLANE UNDER THE ACTION OF FORCES APPLIED TO THE BOUNDARY

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By the method of Fourier integral transforms we construct the exact solution of the problem of equilibrium of a nonhomogeneous half-plane $z \geqslant 0$ under the action of normal and tangential forces applied to the boundary. The shear modulus of the half-plane is a power function of a linear binomial in the Cartesian coordinate z while Poisson ratio is constant.

In the papers [1 – 4], devoted to similar problems, the equilibrium of a half-plane and a half-space $z \geqslant 0$ with modulus of elasticity $E(z) = E_0 z^k$, was investigated. It is obvious that such media are physically not real, since the modulus of elasticity is equal to zero on the surface. This circumstance, in particular, implies a restriction on the possible values of the exponent k. Thus, for example, the formulation of the problem on the action of a distributed load has sense only for $0 \leqslant k < 1$, which in turn, restricts considerably the sphere of applicability of the power law adapted by the authors as an interpolation formula